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Stochastic Processes of a Quantum State

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Starting from a quantum state represented by its wave function $\Psi(x)$, satisfying the Schrödinger equation, we determine stochastic processes which provide the same time evolution for the probability density $\rho(x)=|\Psi(x)|^2$. The transition probabilities of these processes are explicitly built in two circumstances: in the general case, but in an expansion in the time difference, and exactly, but for Gaussian processes. This allows us to discuss the correspondence between quantum states and stochastic processes, which appears not to be one-to-one, but, on the contrary, to associate with the same state an infinity of processes which differ in the fluctuation correlations of the random variable.

1. INTRODUCTION

Besides the operator representation of quantum mechanics (Von Neumann, 1955), now become usual, another representation has also developed, in terms of stochastic processes (Fenyes, 1952; Nelson, 1966). Certainly the absence of a convincing relativistic extension and the lack of representation in the momentum variable make its domain of application remain rather restricted. Nonetheless, the stochastic representation is not without interest: In particular, it allows one to extend the realm of certain classical concepts, like the notion of trajectory, and also, by giving back to the quantum probabilities an ordinary status, to make use of techniques which have been recently developed in this domain (Ito and McKean, 1965; Jona-Lasinio et al., 1982).

The point of view usually adopted tends to substitute the stochastic representation for the ordinary operator representation, and naturally aims at getting rid of the latter. Here, we shall take the opposite point of view: Assuming that a quantum state and its time evolution are correctly and completely described by its wave function and the Schrödinger equation,

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we shall construct one, or several, stochastic processes which lead to the same predictions as quantum mechanics. Such an approach will have a double result: First, to concretely exhibit, by giving their transition probabilities, the stochastic processes one can associate with a quantum state, the existence of which is assured by the stochastic differential and integral calculus (Yasue, 1982); then to analyze and discuss, from the side of quantum mechanics, the hypotheses one must make in order to characterize the processes one wants to associate with quantum states.

It will not be possible to give the exact expressions of the transition probabilities, for any quantum state, but we give them in two complementary cases. A first part gives the transition probabilities for any quantum state (within any potential), but to first orders in an expansion in Δx and Δt, where t and $t+\Delta t$ (x and $x+\Delta x$) are the two times (positions) which enter into the conditional probability $P(x+\Delta x, t+\Delta t|x, t)$. Then, in a second part, an exact expression is given, restricted to Gaussian processes, which nonetheless covers not only the ground state of the harmonic oscillator but also the Gaussian wave packet of the free particle. These two studies then allow one to clarify the correspondence between stochastic processes and wave functions, and to discuss the supplementary assumptions one must make if one insists on the correspondence being one-to-one.

2. EXPANSION OF THE TRANSITION PROBABILITIES

We shall only consider the case of a single degree of freedom, in a one-dimensional space for simplicity, describing the position of a particle, within a potential $V(x)$. All the information on the quantum state of the particle is then contained in its wave function $\Psi(x)$, which corresponds to two real fields: its modulus $|\Psi(x)|^2$ gives the probability density $\rho(x)$ of the observable x, and its phase $S(x)$, with $\Psi=\rho^{1/2}\exp(iS/\hbar)$ provides a velocity field: $\nabla S = mv$. The latter plays an intermediate role in determining the time evolution of the probability density. Indeed, the Schrödinger equation is equivalent to a system consisting of a kinematic equation, or continuity equation:

$$\frac{\partial\rho}{\partial t}+\nabla(v\rho)=0 \qquad \left(\nabla=\frac{\partial}{\partial x}\cdot\Delta=\frac{\partial^2}{\partial x^2}\right) \tag{1}$$

and a dynamical equation:

$$\begin{gathered}\frac{\partial S}{\partial t}+\frac{(\nabla S)^2}{2m}-\frac{\hbar}{2m}\frac{\Delta\rho^{1/2}}{\rho^{1/2}}+V=0 \quad \text{or}\\ \frac{\partial v}{\partial t}+v\nabla v-\frac{\hbar^2}{2m^2}\nabla\left(\frac{\Delta\rho^{1/2}}{\rho^{1/2}}\right)=-\frac{\nabla V}{m}\end{gathered} \tag{2}$$

These equations thus fix the evolution of the probability density in a local way, and at second order (just like the Newton law, thus allowing time reversibility, i.e., invariance under time reflection $t/-t$). It appears better to try to determine the processes, and their transition probabilities, which are likely to represent the time evolution of the probability density, first locally, that is in an expansion with respect to Δt, the time difference. The dependence on a finite number of spatial derivatives only also suggests looking for diffusion processes, that is, the restricted class of those which have only their first two moments of order Δt, the others being of higher order. If $P(x', t'|x, t)$ is the conditional probability, the particle being at x at time t, for the particle to be at x' at time t', these conditions are

$$\begin{aligned} \langle \Delta x \rangle_x &= b(x, t)\Delta t + o(\Delta t) \\ \langle \Delta x^2 \rangle_x &= 2\nu(x, t)\Delta t + o(\Delta t) \\ \langle \Delta x^n \rangle_x &= o(\Delta t); \qquad n > 2 \end{aligned} \tag{3}$$

where

$$\langle \Delta x^n \rangle_x = \int \Delta x^n P(x+\Delta x, t+\Delta t|x, t)\, d\Delta x$$

and $b(x, t)$ and $\nu(x, t)$ stand for the drift and diffusion coefficients. These constraints already allow one to write a general expression (considering the simplest one), for the transition probability:

$$\begin{aligned} &\ln P(x+\Delta x, t+\Delta t|x, t) \\ &\quad = -\frac{1}{2}\ln \alpha\Delta t - \frac{\Delta x^2}{2\alpha\Delta t} + \beta\Delta x + \frac{\gamma}{6}\frac{\Delta x^3}{\Delta t} + \delta\Delta t + \frac{\varepsilon}{2}\Delta x^2 + \frac{\eta}{24}\frac{\Delta x^4}{\Delta t} + o(\Delta t) \end{aligned} \tag{4}$$

where α, β, γ, δ, ε, η depend on x and t.

If they are to correspond to Markov processes, the transition probabilities must satisfy a group composition law, also known as the Chapman-Kolmogorov equation:

$$P(x'', t''|x, t) = \int P(x'', t''|x', t')P(x', t'|x, t)\, dx' \tag{5}$$

In particular, the expansion (4) must satisfy this equation for $t'' = t + \Delta t$, $t' = t + \Delta t/2$. On the other hand, being a probability, P must also be normalized and verify:

$$\int P(x', t'|x, t)\, dx' = 1$$

In the case of locally Gaussian processes ($\gamma = \eta = 0$), it is easy to perform the computations and to obtain, besides the relations induced by (3),

$$\alpha = 2\nu, \qquad \beta = \frac{b}{2\nu}$$

that the stability under convolution is equivalent to the condition:

$$\varepsilon = \frac{1}{2\alpha^2}\frac{\partial\alpha}{\partial t} + \nabla\beta$$

and the normalization to the other condition:

$$\delta = -\frac{\alpha}{2}(\beta^2 + \varepsilon)$$

In the general case, it appears more clever to make use of another equation which is the direct consequence of the previous ones, the Fokker–Planck equation (Schuss, 1980):

$$\frac{\partial P}{\partial t'} + \nabla'(b'P) - \Delta'(\nu' P) = 0 \qquad [b' = b(x', t'),\ \nu' = \nu(x', t')] \tag{6}$$

where $b(x, t)$ and $\nu(x, t)$ are the drift and diffusion fields defined in (3). Replacing P by its expansion in this equation, all its coefficients appear to be determined by the first two moments:

$$\begin{gathered} \alpha = 2\nu, \qquad \beta = \frac{b}{2\nu} - \frac{3}{4}\frac{\nabla\nu}{\nu}, \qquad \gamma = \frac{3}{2}\frac{\nabla\alpha}{\alpha^2}, \qquad \eta = 2\frac{\Delta\alpha}{\alpha^2} - \frac{15}{4}\frac{(\nabla\alpha)^2}{\alpha^3} \\ \varepsilon = \frac{1}{2\alpha^2}\frac{\partial\alpha}{\partial t} + \nabla\beta, \qquad \delta = -\frac{\alpha}{2}\left[\frac{\Delta\alpha}{2\alpha} + \beta\left(\beta + \frac{3}{2}\frac{\nabla\alpha}{\alpha}\right) + \varepsilon\right] \end{gathered} \tag{7}$$

Higher-order terms in the expansion would be determined in exactly the same way, from the Fokker–Planck equation, and shown to depend on the drift and diffusion fields only. Note that the use of the backward equation:

$$\frac{\partial P}{\partial t} + b\nabla P + \nu\Delta P = 0$$

instead of the forward one (6), leads to the same results (7).

The probability density ρ which evolves according to this process, also satisfies the same Fokker–Planck equation:

$$\frac{\partial\rho}{\partial t} + \nabla(b\rho) - \Delta(\nu\rho) = 0$$

Consequently, a necessary and sufficient condition for this evolution to be identical to that induced by the continuity equation, linked to the

Schrödinger equation (1), is that the drift field of the process be given by the quantum state through the equation:

$$b = v + \frac{\nabla(\nu\rho)}{\rho} \tag{8}$$

Hence, to any arbitrary choice of the diffusion field, there corresponds a drift field $b(x, t)$, related to the quantum state through the only equation (8), and thus at least one process [satisfying hypotheses (3), (4)], which makes the probability density evolve according to the Fokker-Planck equation (6), and reproduces the time evolution of the Schrödinger equation (v being given by the dynamical equation). This implies the following result. With any quantum state, described by a wave function satisfying the Schrödinger equation, one can associate an infinity of processes, labeled by $\nu(x, t)$, which reproduce correctly the time evolution of the probability density, as predicted by quantum mechanics. It is even possible to explicitly construct the transition probabilities of these processes, order by order in an expansion with respect to the time difference.

A few remarks are in order. Hypotheses (3) and (4), on the nature of the processes, have the advantage of leading to a direct and easy construction of the transition probabilities, and this in a unique way, starting from the drift and diffusion fields. As no supplementary condition was required, this also shows that any arbitrary choice of these two fields leads to a good process. Nonetheless, let us note that hypotheses (3) and (4), although very useful, are not inescapable, and that there are in fact no arguments, inside quantum or classical mechanics, which can justify them. Thus one could envisage processes with no finite moments [for which $b(x, t)$ and $\nu(x, t)$ do not exist] or simply other stable laws than the Gaussian ones used in (4), which would still correctly represent the time evolution of the probability density. In the absence of any physical argument, it is for convenience that we shall restrict ourselves to hypotheses (3), (4) in the following.

3. TRANSITION PROBABILITIES OF GAUSSIAN PROCESSES

There is one favorable case, for which the expansions of the transition probabilities can be summed, and lead to exact expressions: that of locally Gaussian processes [$\gamma = \eta = 0$, the fluctuations can then depend on time only, $\nu(t)$], with a linear drift field, $b(x, t) = a(t)x$. These processes are associated with quantum states, for which the wave function is of the form

$$\Psi(x, t) = \frac{1}{[2\pi\sigma(t)]^{1/4}} \exp\left\{ -\frac{x^2}{4\sigma(t)} + \frac{i}{\hbar}\left[\frac{m}{4}\frac{\sigma'(t)}{\sigma(t)}x^2 + \zeta(t)\right]\right\}$$

$$\rho(x, t) = \frac{1}{[2\pi\sigma(t)]^{1/2}} \exp\left[-\frac{x^2}{2\sigma(t)}\right]$$

The continuity equation is then automatically satisfied:

$$v(x, t) = \frac{\sigma'(t)}{2\sigma(t)} x \qquad \left(\sigma' = \frac{d\sigma}{dt}\right)$$

and the dynamical equation [with $V(x) = V_0 + V_2 x^2$] reduces to the following ordinary differential equations:

$$-\left(\zeta' + \frac{\hbar^2}{4m}\frac{1}{\sigma}\right) = V_0 \qquad \left(\zeta' = \frac{d\zeta}{dt}\right)$$

$$\frac{\hbar^2}{8m}\frac{1}{\sigma^2} - \frac{m}{2}\left[\left(\frac{\sigma'}{2\sigma}\right)' + \left(\frac{\sigma'}{2\sigma}\right)^2\right] = V_2$$

This includes in particular two important physical cases: The ground state of the harmonic oscillator [$V_0 = 0$, $V_2 = m\omega^2/2$, $\sigma(t) = \hbar/2m\omega$] and the Gaussian wave packet of the free particle:

$$V_0 = V_2 = 0, \qquad 2\sigma''\sigma - \sigma'^2 = \frac{\hbar^2}{m^2}$$

$$\sigma(t) = \sigma(t_0) + \frac{\hbar^2}{4m^2}\frac{(t-t_0)^2}{\sigma(t_0)}$$

These processes can be constructed in the following way, which resembles the functional integration methods and has the merit of making connection with the previous expansions. The transition probabilities satisfy the group relation (5) and can thus be obtained through the following limit:

$$P(x', t'|x, t) = \lim_{n\infty} *^n P_i(x_{i+1}, t_{i+1}|x_i, t_i)$$

where each $P_i(x_{i+1}, t_{i+1}|x_i, t_i)$ is given by ($a = \sigma'/2\sigma - \nu/\sigma$)

$$P_i(x_{i+1}, t_{i+1}|x_i, t_i) = \frac{\exp\left(-(x_{i+1} - x_i - a_i x_i \bar{\Delta} t)^2/\{4\nu_i \bar{\Delta} t[1 + (\nu_i'/2\nu_i + a_i)\bar{\Delta} t]\}\right)}{\{4\pi\nu_i \bar{\Delta} t[1 + (\nu_i'/2\nu_i + a_i)\bar{\Delta} t]\}^{1/2}}$$

at first order in $\bar{\Delta} t$ so that

$$\langle x_{i+1} - x_i \rangle_{x_i} = a_i x_i \bar{\Delta} t \qquad [a_i = a(t_i)]$$

$$\langle (x_{i+1} - x_i)^2 \rangle_{x_i} = 2\nu_i \bar{\Delta} t + o(\bar{\Delta} t) \qquad \left[\nu_i = \nu(t_i),\ \nu_i' = \frac{d\nu}{dt}(t_i)\right]$$

$$(x_i = x + i\bar{\Delta} x,\ t_i = t + i\bar{\Delta} t,\ x' - x = n\bar{\Delta} x,\ t' - t = n\bar{\Delta} t)$$

and $P_i(x', t'|x, t)$ satisfies the Fokker-Planck equation (6). As the drifts are linear, it is then possible to make use of the composition law of gaussians:

$$P=\frac{\exp[-(x'-ax)^2/2\sigma]}{[2\pi\sigma]^{1/2}},$$

$$P'=\frac{\exp[-(x''-a'x')^2/2\sigma']}{[2\pi\sigma']^{1/2}}\to P'*P=\frac{\exp[-(x''-a''x)^2/2\sigma'']}{[2\pi\sigma'']^{1/2}}$$

with $a''=a'a$, $\sigma''=\sigma'+a'^2\sigma$, and to obtain

$$P(x_i, t_i|x, t)=\frac{\exp[-(x_i-\bar{a}_i x)^2/2\bar{\sigma}_i]}{[2\pi\bar{\sigma}_i]^{1/2}}$$

$$\bar{a}_i=\prod_{k=1}^{i}(1+a_k\bar{\Delta}t)+o(\bar{\Delta}t),\qquad \frac{\bar{\sigma}_i}{\bar{a}_i^2}=\sum_{k=1}^{i}2\frac{\nu_k}{a_k^2}\bar{\Delta}t+o(\bar{\Delta}t)$$

The infinite limit then provides a Gaussian form for the transition probabilities:

$$P(x', t'|x, t)=\frac{\exp\{-[x'-\bar{a}(t', t)x]^2/2\bar{\sigma}(t', t)\}}{[2\pi\bar{\sigma}(t', t)]^{1/2}}$$

with

$$\bar{a}(t', t)=\exp\left[\int_t^{t'}a(s)\,ds\right]=\left[\frac{\sigma(t')}{\sigma(t)}\right]^{1/2}\exp\left[-\int_t^{t'}\frac{\nu(s)}{\sigma(s)}ds\right]$$
$$\bar{\sigma}(t', t)=\bar{a}(t', t)^2\int_t^{t'}\frac{2\nu(s)}{\bar{a}(s, t)^2}ds=\sigma(t')\left\{1-\exp\left[-2\int_t^{t'}\frac{\nu(s)}{\sigma(s)}ds\right]\right\}\tag{9}$$

which satisfies the Fokker-Planck equation, as can be deduced from

$$\begin{aligned}\frac{\partial_{t'}\bar{a}(t', t)}{\bar{a}(t', t)}&=\frac{\partial_{t'}\bar{\sigma}(t', t)-2\nu(t')}{2\bar{\sigma}(t', t)}=a(t')\\ \frac{\partial_t\bar{a}(t', t)}{\bar{a}(t', t)}&=-a(t),\qquad \frac{\partial_t\bar{\sigma}(t', t)}{\bar{\sigma}(t', t)}=-2\nu(t)a(t', t)^2\end{aligned}\tag{10}$$

Let us note that it would also have been possible to derive $P(x', t'|x, t)$, by assuming a Gaussian form and requiring the Fokker-Planck equation, which leads to relations (10) and to their solution (9), or else, by requiring that $\rho(x, t)$ be preserved by the process, which also leads to (9).

Using the explicit solutions just obtained, one can first verify the validity of the hypotheses (3) which were supposed to hold on the processes:

$$\langle x'-x\rangle_x=[\bar{a}(t',t)-1]x=a(t)x\Delta t+o(\Delta t)$$

$$\langle (x'-x)^2\rangle_x=\bar{\sigma}(t',t)+[\bar{a}(t',t)-1]^2x^2=2\nu(t)\Delta t+o(\Delta t)$$

$$\langle (x'-x)^3\rangle_x=[\bar{a}(t',t)-1]\{3\bar{\sigma}(t',t)+[\bar{a}(t',t)-1]^2x^2\}x=o(\Delta t)$$

Moreover, these processes also correspond to martingales:

$$\frac{\langle x'\rangle_x}{\bar{a}(t',t_0)}=\frac{x}{\bar{a}(t,t_0)}$$

where $\bar{a}$ is the solution of (10). Let us also simply remark that the invariant distribution ρ also plays the role of an attractive point, in the course of time (for infinite times). Indeed, the iterated action of the process on a given initial distribution (Gaussian):

$$\rho^*(x,t_0)=\frac{\exp[-(x-x_0^*)^2/2\sigma_0^*]}{[2\pi\sigma_0^*]^{1/2}}$$

brings it closer to the invariant distribution:

$$\rho^*(x,t)=\int P(x,t|x_0,t_0)\rho^*(x_0,t_0)\,dx_0$$

$$=\frac{\exp\{-[x-x^*(t)]^2/2\sigma^*(t)\}}{[2\pi\sigma^*(t)]^{1/2}}$$

$$x^*(t)=\bar{a}(t,t_0)x_0^*,\qquad \sigma^*(t)=\bar{\sigma}(t,t_0)+\bar{a}(t,t_0)^2\sigma_0^*$$

verifies

$$\frac{x^*(t)}{\sigma(t)^{1/2}}=\frac{x_0^*}{\sigma(t_0)^{1/2}}\exp\{-[\gamma(t)-\gamma(t_0)]\},\qquad \gamma(t)=\int^t\frac{\nu(s)}{\sigma(s)}\,ds$$

$$\frac{\sigma^*(t)-\sigma(t)}{\sigma(t)}=\left[\frac{\sigma_0^*-\sigma(t_0)}{\sigma(t_0)}\right]\exp\{-2[\gamma(t)-\gamma(t_0)]\}$$

The size of the diffusion coefficients $\nu(t)$ thus corresponds, for the different equivalent processes, to different speeds of attraction by the invariant distribution ρ (one should limit oneself to diffusion coefficients which satisfy

$$\int^t\frac{\nu(s)}{\sigma(s)}\,ds\xrightarrow{t\to\infty}+\infty)$$

With the help of the exact expressions, one can also compute the fluctuations and their correlations:

$$\langle [x(t+\Delta t)-x(t)]^2\rangle_{x,t} = 2\nu(t)\Delta t + o(\Delta t)$$

$$\langle [x(t+\Delta t)-\langle x(t)\rangle]^2\rangle = \sigma(t+\Delta t) = \sigma(t) + \frac{\hbar^2}{4m^2}\,\frac{\Delta t^2}{\sigma(t)}$$

The latter have for minimum [when $\sigma(t)$ varies]: $\Delta t\, \hbar/2m$. Note that this result remains valid for any diffusion function $\nu(t)$.

The different kinematic elements of the process can also easily be computed:

$$\left(\frac{D+D^*}{2}\right)x = v(x,t) = \frac{\sigma'}{2\sigma}\,x, \qquad \left(\frac{D-D^*}{2}\right)x = \frac{\nabla(\nu\rho)}{\rho} = -\frac{\nu}{\sigma}\,x$$

$$\left(\frac{D+D^*}{2}\right)^2 x = \left[\left(\frac{\sigma'}{2\sigma}\right)' + \left(\frac{\sigma'}{2\sigma}\right)^2\right]x, \qquad \left(\frac{D-D^*}{2}\right)^2 x = \frac{\nu^2}{\sigma^2}\,x$$

which confirms (the definitions will be recalled in the next part of the paper) that the Schrödinger equation is equivalent to a Newton equation, where the force is given by

$$F = -2V_2 x$$

and the acceleration by

$$a = \left(\frac{D+D^*}{2}\right)^2 x + \lambda\left(\frac{D-D^*}{2}\right)^2 x$$

with

$$\lambda(t)\nu^2(t) = -\frac{\hbar^2}{4m^2}$$

One will have noticed a common feature of the expansions and exact expressions, for the transition probabilities of the processes one can associate with a quantum state: the presence of an arbitrary function, related to the diffusion coefficient of the process, and as a consequence, of an infinity of equivalent processes for the same state. This property is discussed in detail in the next part.

4. NONUNICITY OF THE PROCESSES

The presence of several (if not an infinity) of equivalent processes, from the point of view of quantum mechanics, is puzzling, especially if one wants to identify them with a real physical phenomenon. The crucial question is then: are there supplementary assumptions, which would allow one to limit to a unique process the class associated with a quantum state, and which could be justified by physical arguments, and confronted with experiment?

The way followed by stochastic mechanics (Nelson, 1966), consists in giving a physical interpretation to the diffusion constant ν, and then considering the dynamical equation (2) as an extension of the classical Newton law ($F = ma$, with F the force, m the mass, and a the acceleration). In this approach, the diffusion coefficient is taken to be constant, independent of space and time, and equal to a universal constant: $\hbar/2m$. This indeed assures the unicity of the process [assuming also hypotheses (3), (4)]. Nonetheless, one must admit that, up to now, no experimental fact has come to justify such an assumption. One intuitive argument consists in identifying (Nelson, 1983) the correlations of the fluctuations:

$$\lim_{\Delta t \to 0} \frac{\langle [x(t+\Delta t) - x(t)]^2 \rangle_{x,t}}{\Delta t}$$

and the fluctuations of the random variable themselves:

$$\lim_{\Delta t \to 0} \frac{\langle [x(t+\Delta t) - \langle x(t) \rangle]^2 \rangle}{\Delta t}$$

The latter are given by the probability density only, taken at different times, and are effectively determined by quantum mechanics. In the case of a free particle, they are easily computed, as was done in the previous section, and can be related to the universal constant $\hbar/2m$. But, as can be seen on the explicit example of the Gaussian free wave packet, the argument which identifies them with the correlations in fluctuation, or the diffusion constant, is incorrect. One must unfortunately accept the evidence, that whatever the diffusion coefficients are, all the corresponding processes lead to the same density probabilities, and thus to the same fluctuations of the random variables, even at different times. The previous argument thus cannot be used to fix the value of the diffusion coefficient, and consequently the process. One should also remark that, in the ordinary framework of quantum mechanics, the correlations in fluctuation are not experimentally accessible. Indeed, the latter require two successive measurements of the variable, (here of position), at different times and on the same state. But no such measurements can be done without perturbing the state in between. Hence, the hypotheses of spatial and temporal homogeneity of the diffusion coefficient, and *a fortiori* its universal value $\hbar/2m$ (Davidson, 1979) can find a justification only by leaving the framework of quantum mechanics, and by relying on experimental evidence of a new type.

One merit of stochastic mechanics it to deduce the dynamical equation (2) from the Newton law. This derivation is accomplished in two steps, by first achieving a stochastic extension of classical kinematics, and then by replacing in the Newton law, the acceleration by its stochastic extension. Thus the fluctuations on the position lead one to introduce two different time derivatives, a forward one, $D=(\partial/\partial t)+b\nabla+\nu\Delta$, and a backward one, $D^*=(\partial/\partial t)+b^*\nabla-\nu\Delta$ [where $b^*=b-2\nabla(\nu\rho)/\rho$], which result in two different velocities $b=Dx$, and $b^*=D^*x$, and then in four different accelerations DDx, DD^*x, D^*Dx, D^*D^*x. In order to apply the Newton law, one has then to choose one acceleration among four. Recalling reversibility, that is symmetry under time reversal, one can reduce the choice to a linear combination of only two of them:

$$a=\left(\frac{D+D^*}{2}\right)^2 x+\lambda\left(\frac{D-D^*}{2}\right)^2 x \tag{11}$$

Remarkably enough, for $\lambda=-1$ and $\nu=\hbar/2m$, the Newton equation identifies with the dynamical equation (2). However, this correspondence is not sufficient in itself to justify simultaneously both choices for λ and ν. Indeed, the identification of the Newton law with the dynamical equation occurs as soon as the following relation between λ and ν is satisfied:

$$\lambda=-\left(\frac{\hbar}{2m}\right)^2\frac{\nabla[\rho\nabla(\nabla\rho/\rho)]}{\nabla[\nu\rho\nabla(\nabla(\nu\rho)/\rho)]} \tag{12}$$

Hence, for any diffusion field $\nu(x, t)$ the Schrödinger equation identifies with the system consisting in the continuity equation and the Newton law, if one correctly defines the acceleration a, according to (11) and (12). Sure enough, several objections can be made against such an ad hoc choice of the acceleration. Unfortunately, these do not hold, as we show now. For instance, one could require that the definition of the acceleration be universal and do not depend on the quantum state, as is the case here for λ does depend *a priori* on the probability density ρ. First of all, one should note that even in the case of λ being independent of ρ, the acceleration does still depend on it, through the dependences of D and D^*. But anyway, such a definition could apply to pure quantum states only. Indeed, it has been shown that in the case of mixed quantum states, one can preserve the equivalence between the Schrödinger equation and the Newton law, only at the price of a redefinition of the acceleration, which then depends on the state in an irreducible way (and even, more precisely, on the pure state components of the mixture) (Jaekel and Pignon, 1984). In any case, even a universal definition limited to pure quantum states (which, by the way, would require the introduction of a new concept, alien to classical mechanics), would not be sufficient to fix the value of λ equal to -1. Indeed, there still remain pure temporal dependences for λ and ν, being fixed in a universal way (independent of the state) through the relation:

$$\lambda(t)\nu^2(t) = -\left(\frac{\hbar}{2m}\right)^2$$

This is precisely the situation which has been described in the previous section on the explicit example of the Gaussian wave function.

The problem of nonunicity of the process one can associate with a quantum state, thus reflects itself in the arbitrariness of two choices, which are linked together (of diffusion coefficient and of acceleration). Moreover, both quantities are related to the time correlations of the variable (here the position) and cannot be obtained, in the framework of quantum mechanics, without being destroyed.

5. CONCLUSION

It is thus possible to associate with each quantum state, that is with the same wave function, a large infinity (corresponding to an arbitrary function of space and time) of different stochastic processes, which all lead to the same time evolution for the probability density. On the other hand, the wave function constitutes, up to now, all the information one needs on the quantum state to account for all the experimental facts which depend on the latter. Unless experiments of a new type, like a (even indirect) measurement of the fluctuation correlations of the process are made, which would go beyond description by quantum mechanics, nothing can allow one to distinguish one process among the others, and to give it the status of real process. Presently, one can just consider all these processes on the same footing, and consequently, as virtual processes (maybe, like those occurring in Feynman diagrams).

We have not insisted on the hypotheses which bear on the existence of the first two moments of the transition probability. The impossibility of physically justifying these hypotheses, moreover indicates that the class of processes one can associate with a quantum state, could be even larger than the one exhibited here. Such a proliferation of the processes looks disturbing. One way to escape it, would be to isolate universal properties, which are shared by all the processes, and to relate them to the wave function of the quantum state. The attraction property shown in part 3 by the probability density, with its time evolution, might not be restricted to Gaussian cases and would then reinforce the idea that the quantum wave function describes the universal limit of a large class of conservative processes.

REFERENCES

Davidson, M. (1979). *Letters in Mathematical Physics*, **3**, 271.

Fenyes, I. (1952). Zeitschrift für Physik, **132**, 81.

Ito, K. and MacKean, H. (1965). *Diffusion Processes and Their Sample Path.* Springer-Verlag, Berlin.

Jaekel, M. T., and Pignon, D. (1984). *Journal of Physics A*, **17**, 131.

Jona-Lasinio, G., Martinelli, F., and Scoppola, E. (1982). *Mathematical Problems in Theoretical Physics*, Proceedings of the VIth International Conference in Mathematical Physics. Springer-Verlag, Berlin.

Nelson, E. (1966). *Physical Review*, **150**, 1079.

Nelson, E. (1983). *VIIth International Congress in Mathematical Physics.* North-Holland, Amsterdam.

Schuss, Z. (1980). *Theory and Applications of Stochastic Differential Equations.* Wiley & Sons, New York.

Von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics.* Princeton University Press, Princeton, New Jersey.

Yasue, K. (1982). *Journal of Mathematical Physics*, **23**, 1577.